CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter describes the detail techniques applied for solving linear and nonlinear partial differential equations in tumour growth model. The methods used are Adomian Decomposition Method (ADM) and Homotopy Perturbation Method (HPM). These methods produced approximate analytical solution in a series form. Prior to this, most of the tumours models are being solved numerically using finite difference method.

3.2 Adomian Decomposition Method (ADM)

In the 1980's, George Adomian introduced a new method to solve nonlinear functional equation (Adomian, 1988). This method known as the Adomian Decomposition Method (ADM) and has been the subject of much investigation (Babolian & Biazar, 2002; Cherruault *et.al*, 1995; Lesnic, 2002). This method produces a convergent series solution (Hashim, 2006) and the truncated series provides an approximate solution. The issue of convergence is addressed by several researches (Babolian & Biazar, 2002; Cherruault *et.al*, 1995; Lesnic, 2002). According to Cherruault *et.al.*, (1995), the series produced by the decomposition method is absolutely convergent as well as uniformly convergent. This is the case because the series rearranges a strongly convergent Taylor series of the analytic functions u and $f(u)$. The series converges uniformly (and absolutely and in norm), hence the sum is not changed by rearrangement of the terms (Cherruault *et.al*, 1995). Babolian & Biazar (2002) provide a definition from which the order of convergence for the method could be determined. Of course, having a higher order of convergence is desirable since then the series will converges more rapidly.

The ADM involves separating the equation under investigation into linear and nonlinear portions. The linear operator representing the linear portion of the equation is inverted and the inversed operator is then applied to the equation. Any given conditions are taken into consideration. The nonlinear portion is decomposed into a series of Adomian polynomials. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials.

In reviewing the basic methodology involved, a general nonlinear differential equation will be used for simplicity. Consider

$$Fu = g(t) \tag{3.1}$$

where F is a nonlinear differential operator and u and g are function of t. Let's begin by rewritting the equation in the operator form

$$Lu + Ru + Nu = g \tag{3.2}$$

where L is an operator representing the linear portion of F which is easily invertible, R is a linear operator for the remainder of the linear portion and N is a nonlinear operator representing the nonlinear terms in F. Applying the inverse operator L^{-1}, the equation then becomes

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu \tag{3.3}$$

Since F is taken to be a differential operator and L is linear, L^{-1} would represent an integration and with any given initial or boundary conditions, $L^{-1}Lu$ will give an equation for u incorporating these conditions. This gives

$$u(t) = g(t) - L^{-1}Ru - L^{-1}Nu \tag{3.4}$$

where $g(t)$ represent the function generated by integrating g and using the initial and boundary conditions. Next, let's assume that the unknown function can be written as an infinite series

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{3.5}$$

We set $u_0 = f(t)$ and the remaining terms are to be determined by a recursive relationship defined later. This is found by first decomposing the nonlinear term into series of Adomian polynomials, A_n. The nonlinear term is written as

$$Nu = \sum_{n=0}^{\infty} A_n \tag{3.6}$$

In order to determine the Adomian polynomials, a grouping parameter, λ is introduced. It should be noted that λ is not a "smallness parameter" (Cherruault *et.al*, 1995). The series

$$u(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n \tag{3.7}$$

and

$$Nu(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n \tag{3.8}$$

are established. Then A_n can be determined by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} Nu(\lambda) \big|_{\lambda=0} \tag{3.9}$$

From

$$\sum_{n=0}^{\infty} u_n = u_0 - L^{-1} \sum_{n=0}^{\infty} Ru_n - L^{-1} \sum_{n=0}^{\infty} A_n \tag{3.10}$$

the recursive relationship is found to be

$$u_0 = f(t) \tag{3.11}$$

$$u_{n+1} = L^{-1}Ru_n + L^{-1}A_n \tag{3.12}$$

3.3 Homotopy Perturbation Method (HPM)

The homotopy perturbation method (HPM) first proposed by Ji Huan He (1999) and was further developed and improved by He (2006). He developed and formulated HPM by merging the standard homotopy and perturbation. HPM proved to be compatible with the versatile nature of the physical problems and has been applied to a wide class of functional equations (He, 2005). In this technique, the solution is given in an infinite series usually converging to an accurate solution. It is worth mentioning that HPM is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The HPM is applied for all the nonlinear terms in the problem without discretizing either by finite difference or by spline techniques at the nodes and involves laborious calculations coupled with a strong possibility of the ill conditioned resultant equations which are a complicated problem to solve. Moreover, unlike the method of separation variables that requires initial and boundary conditions, HPM provides an analytical solution by using the initial condition only. The fact that HPM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method. Ghorbani (2009) introduced He's polynomials coupled with the conclusion that He's polynomials are easier to calculate, are more user friendly and are independent of the complexities arising in calculating the so-called Adomian's polynomials. It is to be highlighted that He's polynomials are calculated from HPM. Here the technique will be used to construct a perturbation equation. To illustrate its basic ideas, we consider the following non-linear algebraic equation:

$$f(x) = 0, \qquad x \in \Re. \tag{3.13}$$

We construct a homotopy $\Re \times [0,1] \to \Re$ which satisfies

$$\mathrm{H}(\xi, p) = pf(\xi) + (1-p)[f(\xi) - f(x_0)] = 0, x \in \Re, p \in [0,1] \tag{3.14a}$$

or

$$\mathrm{H}(\xi, p) = f(\xi) - f(x_0) + pf(x_0) = 0, x \in \Re, p \in [0,1], \tag{3.14b}$$

where p is an embedding parameter. x_0 is an initial approximation of Eq. (3.13)

It is obvious that

$$\mathrm{H}(\xi, 0) = f(\xi) - f(x_0) = 0, \tag{3.15}$$

$$\mathrm{H}(\xi, 1) = f(\xi) = 0, \tag{3.16}$$

the changing process of p from zero to unity is just that of $\mathrm{H}(\xi, p)$ from $f(\xi) - f(x_0)$ to $f(\xi)$. In topology, this is called deformation, and $f(\xi) - f(x_0)$, $f(\xi)$ are called homotopic.

Due to the fact that $0 \le p \le 1$, so the embedding parameter can be considered as a small parameter. Applying the perturbation techniques, we can assume that the solution of Eqs. (3.14a) and (3.14b) as

$$\xi = \xi_0 + p\xi_1 + p^2\xi_2 + p^3\xi_3 + \ldots\ldots. \tag{3.17}$$

To obtain its approximate solution of Eqs. (3.14a) and (3.14b), we, at first, expand $f(\xi)$ into a Taylor series

$$f(\xi) = f(\xi_0) + f^{'}(\xi_0)(p\xi_1 + p^2\xi_2 + \ldots..) + \frac{1}{2!} f(\xi_0)(p\xi_1 + p^2\xi_2 + \ldots..)^2 + \ldots\ldots \tag{3.18}$$

Substituting Eq. (3.18) into Eqs. (3.14a) and (3.14b) and equating the coefficients of like powers of

p , we obtain

$$p^0 : f(\xi_0) - f(x_0) = 0, \tag{3.19}$$

$$p^1 : f'(\xi_0)\xi_1 + f(x_0) = 0, \tag{3.20}$$

$$p^2 : f'(\xi_0)\xi_2 + \frac{1}{2!} f''(\xi_0)\xi_1^{\ 2} = 0. \tag{3.21}$$

From Eq. (3.20), ξ_1 can be solved as

$$\xi_1 = -\frac{f(x_0)}{f'(\xi_0)}. \tag{3.22}$$

If its first-order approximation is sufficient, then we have

$$\xi = \xi_0 - \frac{pf(\xi_0)}{f'(\xi_0)}. \tag{3.23}$$

The substitution $p = 1$ in Eq. (3.23) yields the first-order approximate solution of Eqs. (3.14a) and (3.14b).

$$x = \xi_0 - \frac{f(\xi_0)}{f'(\xi_0)}. \tag{3.24}$$

Using Eq. (3.24) as an initial approximation in Eq. (3.13) repeatedly, we have the following iteration formula:

$$x_{n+1} = \xi_n - \frac{f(\xi_n)}{f'(\xi_n)}. \tag{3.25}$$

From Eq. (3.19), we can obtain one of its solutions $\xi_0 = x_0$, under this condition, iteration formula (3.25) can be re-written down as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3.26}$$

which is the well-known Newton iteration formula.

Using the same manipulation, from Eq. (3.26), ξ_2 can be solved, and the following iteration formula can be obtained:

$$x_{n+1} = \xi_n - \frac{f(\xi_n)}{f'(\xi_n)} - \frac{f''(\xi_n)}{2f'(\xi_n)}\left\{\frac{f(\xi_n)}{f'(\xi_n)}\right\}^2. \tag{3.27}$$

The iteration formula (3.27) is called the Newton-like iteration formula with second-order approximation. The approximate solution obtained by the above iteration formula (3.24) converges to its solution faster than the Newton iteration formula (3.25).

Both methods yield rapidly convergent series solutions for linear and nonlinear equations. The advantages of these methods are that they provide direct scheme for solving the problem, i.e. without the need for linearization and discretization. These methods also eliminate the difficulties and massive computation work.